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UNIFORM LIMIT THEOREMS FOR SYNCHRONOUS PROCESSES
WITH APPLICATIONS TO QUEUES

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Uniform Limit Theorems For Synchronous Processes
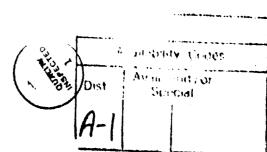# With Applications To Queues

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Abstract

Let $X = \{X(t) : t \geq 0\}$ be a positive recurrent synchronous process (PRS), that is, a process for which there exists an increasing sequence of random times $\tau = \{\tau(k)\}$ such that for each k the distribution of $\theta_{\tau(k)}X \stackrel{def}{=} \{X(t + \tau(k)) : t \geq 0\}$ is the same and the cycle lengths $T_n \stackrel{def}{=} \tau(n+1) - \tau(n)$ have finite first moment. In the present paper we investigate conditions under which the Cesaro averaged functionals $\overline{\pi}_t(f) \stackrel{def}{=} \frac{1}{t}\int_0^t E(f(\theta_s X))ds$ converge uniformly (over a class of functions) to $\pi(f)$, where π is the stationary distribution of X. We show that $\overline{\pi}_t(f) \longrightarrow \pi(f)$ uniformly over f satisfying $\|f\|_\infty \leq 1$ (total variation convergence). We also show that to obtain uniform convergence over all f satisfying $|f| \leq g$ ($g \in L_1^+(\pi)$ fixed) requires placing further conditions on the PRS. This is in sharp contrast to both classical regenerative processes and discrete time Harris recurrent Markov chains (where renewal theory can be applied) where such uniform convergence holds without any further conditions. For continuous time Harris recurrent Markov processes (where renewal theory can not be applied) we show that these further conditions are in fact automatically satisfied. In this context, applications to queueing models are given.

Key Words: Synchronous Process, Cesaro convergence, Limit Theorems, Point Processes

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1. Preliminaries and Introduction

Throughout this paper, $X = \{X(t) : t \geq 0\}$ will denote a stochastic process taking values in a complete separable metric state space S and having paths in the space $\mathcal{D} = \mathcal{D}_S[0,\infty)$ of functions $f : R_+ \to S$ that are right continuous and have left hand limits. $\mathcal{D}$ is endowed with the Skorohod topology (and is a complete separable metric space). $(\Omega, \mathcal{F}, P)$ will denote the underlying probability space and we view X as a random element of $\mathcal{D}$. Let Δ denote an arbitrary fixed element not in the set S. We then endow $\bar{S} \overset{\text{def}}{=} S \cup \{\Delta\}$ with the one-point compactification topology.

Definition 1.1. *X is said to be a* **synchronous** *process with respect to the random times $0 \leq \tau(0) < \tau(1) < \cdots$ (with $\lim_{n\to\infty} \tau(n) = \infty$ a.s.) if $\{X_n : n \geq 1\}$ forms a stationary sequence in the space $\mathcal{D}_{\bar{S}}^\infty$, where*

$$X_n(t) = \begin{cases} X(\tau(n-1) + t), & \text{if } 0 \leq t < T_n; \\ \Delta, & \text{if } t \geq T_n. \end{cases}$$

$T_n \overset{\text{def}}{=} \tau(n) - \tau(n-1)$ is called the nth cycle length, X_n is called the nth cycle and we refer to $(\tau(n))$ as the synch-times for X with counting process $N(t) = $ the number of synch times that fall in the interval $[0, t]$.

Definition 1.2. *A synchronous process X is called* **non-delayed** *if $\tau(0) = 0$ a.s.;* delayed *otherwise. It is called* **positive recurrent** *if $E(T_1) < \infty$;* **null recurrent** *otherwise. It is called* **ergodic** *if it is positive recurrent and the invariant σ-field, $\mathcal{I}$, of $\{X_n, T_n\}$ is trivial. $\lambda \overset{\text{def}}{=} \frac{1}{E(T_1)}$ is called the* **rate** *of the synch times. $\hat{\lambda} \overset{\text{def}}{=} \frac{1}{E(T_1|\mathcal{I})}$ is called the* **conditional rate.**

From now on, PRS will be used to abbreviate *positive recurrent synchronous process*. To help the reader, an appendix is included at the end of this paper giving a brief introduction to PRS's.

$\theta_t : \mathcal{D} \to \mathcal{D}$ denotes the *shift operator* $(\theta_t x)(s) = x(t + s)$, P^0 denotes the probability measure under which X is non-delayed; $P^0(A) = P(\theta_{\tau(1)} X \in A)$ and P^* denotes the probability measure under which X has the stationary distribution π (see the Appendix).

The important point here is that at the random times $\tau(k)$, $X(t)$ and its future probabilistically start over. However, in contrast to classical regenerative processes (CRP's), or the regenerative structure found in Harris recurrent Markov chains (HRMC's), the future is not necessarily independent of any of the past $\{\tau(1), \ldots, \tau(k); X(s) : 0 \leq s \leq \tau(k)\}$. In particular τ does not (in general) form a renewal process

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and hence renewal theory does not apply to synchronous processes. Natural questions arise, however, as to which of known limit theorems etc. that hold for CRP's and HRMC's actually do not depend upon renewal theory and can in fact be extended to cover PRS's. One such result was given recently in [4] where it was shown that for a PRS, the distributions of $\theta_s \circ X \overset{\text{def}}{=} \{X(s+t) : t \geq 0\}$ are in fact tight in the function space $\mathcal{D}$. But what about limit theorems for PRS's? For example, although $N(t)/t \longrightarrow \hat{\lambda}$ P a.s. as $t \longrightarrow \infty$, what can be said about

$$E(N(t))/t \longrightarrow E\hat{\lambda}, \tag{1.1}$$

which *does* hold true for a renewal process? Similarly, for an ergodic PRS, although $\frac{1}{t}\int_0^t f(\theta_s \circ X)ds \longrightarrow \pi(f)$, P^0 a.s. for any $f \in L_1(\pi)$, what can be said about

$$\overline{\mu}_t(f) \overset{\text{def}}{=} \frac{1}{t}\int_0^t E^0 f(\theta_s \circ X)ds \longrightarrow \pi(f), \quad f \in L_1(\pi), \tag{1.2}$$

or

$$\sup_{\|f\|_\infty \leq 1} |\overline{\mu}_t(f) - \pi(f)| \longrightarrow 0, \tag{1.3}$$

or (more generally)

$$\sup_{|f| \leq g} |\overline{\mu}_t(f) - \pi(f)| \longrightarrow 0, \quad \text{for each} \quad g \in L_1^+(\pi), \tag{1.4}$$

all three of which holds true for CRP's and HRMC's?

We show that (1.3) is always true for a PRS (ergodic or not), whereas (1.1), (1.2) and (1.4) require extra conditions (even in the ergodic case). These extra conditions turn out to be automatically satisfied for continuous time Harris recurrent Markov processes (HRMP's). In this context we give some applications to queueing models.

2. Limit theorems for $N(t)$

In this section we present counterexamples showing that (1.1) is false in general. In fact we show that even in the ergodic case it is possible that $E^0 N(t) = \infty$.

Let $\tau = \{\tau(n)\}$ be the synch times of a non-delayed PRS X. Let $N(t)$ denote the corresponding counting process. Under P^0, X is non-delayed and the point process τ is called a *Palm* version in which case $\{T_n\}$ forms a stationary sequence. Under P^*, X is stationary as is the point process τ (see for example, [7]).

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Example (1) Let Z be a r.v. such that $P(Z \geq 1) = 1$ and $E(Z) = \infty$. Define $T_n = 1/Z$ ($n \geq 1$); thus $\tau(n) = n/Z$ and $E(T_n) \leq 1 < \infty$. Observe that $P(N(t) > n) = P(\tau(n) < t) = P(tZ > n)$ so that indeed $E(N(t)) = \infty$ for all $t > 0$. Observe, however, that $\{T_n\}$ is not ergodic; its invariant σ-field is precisely $\sigma(Z)$.

Whereas Example (1) is not ergodic, our next example is.

Example (2) Consider a discrete time renewal process with cycle length distribution $\mathcal{P} = \{p_k : k \geq 1\}$ having finite and non-zero first moment, $1/\mu$, but infinite second moment. Let $B(n)$ denote the corresponding discrete time forward recurrence time process. B is a positive recurrent Markov chain with invariant probability distribution $\alpha_n \overset{\text{def}}{=} \mu \sum_{k \geq n} p_k$ (the *equilibrium* distribution of $\mathcal{P}$). α has infinite first moment; $\sum_{k=1}^{\infty} k\alpha_k = \infty$. Let $h(k) = 1/(k^2 + 1)$ and define a point process by $T_n = h(B(n))$; $\tau(n) = T_1 + T_2 + \cdots + T_n$. Observe that $0 < T_n \leq 1$ for all n. Under α, $\{T_n\}$ is a stationary ergodic sequence and hence corresponds to a Palm version with $0 < \lambda \overset{\text{def}}{=} \{E_\alpha h(B(0))\}^{-1} < \infty$. It is also positive recurrent regenerative; it regenerates whenever $B(n) = 0$. Let $\gamma = min\{n \geq 0 : B(n) = 0\}$, and observe that $\gamma = B(0)$. Let $M = \sum_{i=0}^{\infty} h(i)$. Consider the random time $T = \tau(\gamma) = T_1 + \cdots + T_\gamma$. Observe that $T \leq M < \infty$ and that $N(T|X_0 = k) = k$. Thus for $t \geq M$ we obtain

$$
\begin{aligned}
E^0 N(t) &= \sum_{k=1}^{\infty} E^0\{N(t)|B(0) = k\}\alpha_k \\
&\geq \sum_{k=1}^{\infty} E^0\{N(T)|B(0) = k\}\alpha_k \\
&= \sum_{k=1}^{\infty} k\alpha_k = \infty.
\end{aligned}
$$

The important point here is that in general, $\{N(t)/t : t \geq 0\}$ is not uniformly integrable (UI). We do, however, have some sufficient conditions.

Proposition 2.1. *Suppose $0 < \lambda^{-1} = E^0(T_1) < \infty$. If either there exists an $\epsilon > 0$ such that $P^0(T_1 > \epsilon) = 1$, or the interevent times $\{T_n\}$ form a k-dependent process, then (1.1) holds.*

Proof: Suppose $P^0(T_1 > \epsilon) = 1$. Then for all $t \geq 0$, $\frac{N(t)}{t} \leq \frac{1}{\epsilon} + 1$, P a.s. (even in the delayed case) and hence is UI. Suppose now that the T_n are k-dependent, that is, for each n, $\{T_{n+j} : j \geq k\}$ is independent of $\{T_m : m \leq n\}$. It follows that for each i ($0 \leq i \leq k - 1$) $T(i) = \{T_{kn+i} : n \geq 1\}$ defines a (possibly delayed) renewal process. Let $N^{(i)}(t)$ denote the corresponding i^{th} counting process. Clearly

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$N(t) \leq N^{(0)}(t) + N^{(1)}(t) + \cdots + N^{(k-1)}(t)$. Since $\frac{N^{(i)}(t)}{t}$ is UI for each i (because, for each i, it is from a renewal process), so is $\frac{N(t)}{t}$. $\blacksquare$

Remark (2.1): By changing our example (2) slightly, we can actually obtain a null recurrent version: Let $\{T_n\}$ and $N(t)$ be from example (2) (under α), and let H_1 denote the distribution of T_n. Let $\{L_k\}$ be non-negative i.i.d. $\sim H_2$, with H_2 having infinite first moment. Define $\gamma_0 = 0$, $\gamma_{k+1} = min\{n > \gamma_k : B(n) = 0\}$. Between $T_{\gamma_k - 1}$ and T_{γ_k}, insert L_k. The idea here is to start off the kth regenerative cycle with L_k and then proceed as before. This gives rise to a new sequence of interevent times $\tilde{T}_n$. Taking a Palm version of this new point process yields a stationary ergodic sequence $\tilde{T}_n^0$ such that $E(\tilde{T}_n^0) = \infty$; $\tilde{T}_n^0 \sim (1-p)H_1 + pH_2$ where $p = 1/(1 + 1/\mu)$. Letting $\tilde{N}(t)$ denote the associated counting process, we obtain $E(\tilde{N}(t)) \geq (1 - p)E(N(t)) = \infty$ for $t \geq M$.

Remark (2.2): In our example(2), it is true, however (as is well known more generally in the point process literature), that $E^*(N(t)) = \lambda t$ for all $t \geq 0$ and hence that the *intensity* $\stackrel{\text{def}}{=} E^*(N(1))$ is finite and is equal to λ. It is only the Palm version that can blow up.

3.1 Uniform limit theorems for X

We first present an example of an ergodic PRS together with an $f \in L_1(\pi)$, such that $\bar{\mu}_t(f) = \infty$. In particular, (1.2) does not hold.

Example (3) Consider T_n and $B(n)$ from example (2). Form a semi-Markov process $X(t)$ by using T_n as the holding time for B_n. Then for $B(0) \sim \alpha$, X is an ergodic PRS with synch times $\tau(n)$. Now choose an $f \geq 0$ such that $fh \in L_1(\alpha)$ but

$$\sum_i f(i)h(i) \sum_{k \geq i} \alpha_k = \infty.$$

Then

$$E^0 \int_0^{\tau(1)} f(X(s))ds = \sum_k E^0 \{ \int_0^{\tau(1)} f(X(s))ds; X(0) = k \}$$
$$= \sum_k E^0 \{ f(k)h(k); X(0) = k \}$$
$$= \sum_k f(k)h(k)\alpha_k < \infty.$$

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Hence $f \in L_1(\pi)$. On the other hand, for $t \geq M$,

$$t\overline{\mu}_t(f) = E^0 \int_0^t f(X(s))ds \geq E^0 \int_0^T f(X(s))ds$$

$$= \sum_k E^0\{\int_0^T f(X(s))ds; X(0) = k\}$$

$$= \sum_k E^0\{\sum_{j=0}^k f(k-j)h(k-j); X(0) = k\}$$

$$= \sum_k \alpha_k \sum_{j=0}^k f(j)h(j)$$

$$= \sum_{j=0}^\infty f(j)h(j) \sum_{k \geq j} \alpha_k = \infty.$$

We do, however, have the following

Theorem 3.1. *If X is PRS and $g \in L_1^+(\pi)$ such that $\frac{1}{t}E \int_0^{t \wedge \tau(0)} g(\theta_s X)ds \longrightarrow 0$ and $\{\frac{1}{t}\int_0^t g(\theta_s \circ X)ds :$*

$t \geq 0\}$ is UI under P^0, then

$$\sup_{|f| \leq g} |\overline{\mu}_t(f) - \pi(f)| \longrightarrow 0. \tag{3.1}$$

Before proving Theorem 3.1 we state an important corollary obtained immediately by using the function

$g \equiv 1.$

Corollary 3.1. *If X is a PRS then $\overline{\mu}_t$ converges to π in total variation.*

Proof of Theorem 3.1: Assume at first that $0 \leq f \leq g$ and that X is non-delayed. For $\epsilon > 0$ let $\mathcal{A}(\epsilon, t)$

denote the event $\{N(t) \geq (\hat{\lambda} + \epsilon)t\}$. Let $J_n = J_n(f) \overset{\text{def}}{=} \int_{\tau(n-1)}^{\tau(n)} f(\theta_t \circ X)dt$. Let $E_\mathcal{I}$ denote E^0 conditional

on the invariant σ-field $\mathcal{I}$. Then

$$\frac{1}{t}E_\mathcal{I} \int_0^t f(\theta_s X)ds = \frac{1}{t}E_\mathcal{I}\{\int_0^t f(\theta_s X)ds; \mathcal{A}(\epsilon, t)^c\} + \frac{1}{t}E_\mathcal{I}\{\int_0^t f(\theta_s X)ds; \mathcal{A}(\epsilon, t)\}$$

$$\leq [\frac{(\hat{\lambda} + \epsilon)t + 1}{t}]E_\mathcal{I}J_1 + \frac{1}{t}E_\mathcal{I}\{\int_0^t g(\theta_s X)ds; \mathcal{A}(\epsilon, t)\} \tag{3.2}$$

$$\leq \hat{\lambda}E_\mathcal{I}J_1 + (\epsilon + \frac{1}{t})E_\mathcal{I}J_1 + \frac{1}{t}E_\mathcal{I}\{\int_0^t g(\theta_s X)ds; \mathcal{A}(\epsilon, t)\}.$$

Taking expectations in (3.2) with respect to E^0 yields

$$\frac{1}{t}E^0 \int_0^t f(\theta_s X)ds \leq \pi(f) + (\epsilon + \frac{1}{t})E^0 J_1 + \frac{1}{t}E^0\{\int_0^t g(\theta_s X)ds; \mathcal{A}(\epsilon, t)\}. \tag{3.3}$$

By the uniform integrability hypothesis, the last term in (3.3) tends to zero. Moreover, ϵ was arbitrary. We

thus obtain

$$\overline{\lim_{t \to \infty}} \sup_{f \leq g} \{\overline{\mu}_t(f) - \pi(f)\} \leq 0. \tag{3.4}$$

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In a similar manner we obtain a lower bound: For $\epsilon > 0$

$$\frac{1}{t}E_{\mathcal{I}}\int_0^t f(\theta_s X)ds \geq \frac{1}{t}E_{\mathcal{I}}\{\sum_{k=1}^{\lceil(\hat{\lambda}-\epsilon)t\rceil} J_k; \mathcal{A}(-\epsilon,t)\}$$

$$\geq (\frac{(\hat{\lambda}-\epsilon)t-1}{t})E_{\mathcal{I}}J_1 - \frac{1}{t}E_{\mathcal{I}}\{\sum_{k=1}^{\lceil(\hat{\lambda}-\epsilon)t\rceil} J_k(g); \mathcal{A}(-\epsilon,t)^c\}$$

$$\geq \hat{\lambda}E_{\mathcal{I}}J_1 - (\epsilon + \frac{1}{t})E_{\mathcal{I}}J_1 - \frac{1}{t}E_{\mathcal{I}}\{\sum_{k=1}^{\lceil(\hat{\lambda}-\epsilon)t\rceil} J_k(g); \mathcal{A}(-\epsilon,t)^c\},$$

which after taking expectations yields

$$\overline{\mu}_t(f) - \pi(f) \geq -(\epsilon + \frac{1}{t})E^0 E_{\mathcal{I}}J_1 - \frac{1}{t}E^0\{\sum_{k=1}^{\lceil\hat{\lambda}t\rceil} J_k(g); \mathcal{A}(-\epsilon,t)^c\}. \tag{3.5}$$

Since $g \in L_1(\pi)$, $\frac{1}{t}\sum_{k=1}^{\lceil\hat{\lambda}t\rceil} J_k(g)$ is UI since it converges $a.s.$ to $\hat{\lambda}E_{\mathcal{I}}J_1(g)$ and has mean, $E^0\{\frac{\lceil\hat{\lambda}t\rceil}{t}E_{\mathcal{I}}J_1(g)\}$,

for each t. Thus the last term in (3.5) tends to zero. Consequently

$$\varlimsup_{t\to\infty}\sup_{f\leq g}\{\pi(f) - \overline{\mu}_t(f)\} \leq 0, \tag{3.6}$$

and we thus obtain (3.1). The case of f with arbitrary sign can be handled similarly; we leave out the details.

In the delayed case, we have on the one hand that

$$\frac{1}{t}E\int_0^t f(\theta_s \circ X)ds \leq \frac{1}{t}E\int_0^{t\wedge\tau(0)} g(\theta_s X)ds + \frac{1}{t}E^0\int_0^t f(\theta_s \circ X)ds. \tag{3.7}$$

The first term on the rhs tends to zero by assumption, thus, giving the necessary upper bound. On the other hand, for $t \geq M \geq 0$

$$E\int_0^t f(\theta_s \circ X)ds = E\{\int_0^t f(\theta_s \circ X)ds; \tau(0) \leq M\} + E^0\{\int_0^t f(\theta_s \circ X)ds; \tau(0) > M\}$$

$$\geq E\{\int_{\tau(0)}^t f(\theta_s \circ X)ds; \tau(0) \leq M\}$$

$$\geq E^0\{\int_0^{t-M} f(\theta_s \circ X)ds; \tau(0) \leq M\} \tag{3.8}$$

$$\geq E^0\{\int_0^{t-M} f(\theta_s \circ X)ds\} - E^0\{\int_0^t g(\theta_s \circ X)ds; \tau(0) > M\}$$

$$\geq E^0\{\int_0^t f(\theta_s \circ X)ds\} - E^0\{\int_{t-M}^t g(\theta_s \circ X)ds\} - E^0\{\int_0^t g(\theta_s \circ X)ds; \tau(0) > M\}.$$

Using in $M = \epsilon t$, $0 < \epsilon < 1$, in (3.8) yields

$$\frac{1}{t}E\int_0^t f(\theta_s \circ X)ds \geq \frac{1}{t}E^0\{\int_0^t f(\theta_s \circ X)ds\} - \frac{1}{t}E^0\{\int_{t-t\epsilon}^t g(\theta_s \circ X)ds\} - \frac{1}{t}E^0\{\int_0^t\{g(\theta_s \circ X)ds; \tau(0) > \epsilon t\}\}. \tag{3.9}$$

The last integral above tends to zero by the UI assumption under P^0. The middle integral converges to

$\epsilon E^0\{\hat{\lambda}E_{\mathcal{I}}J_1(g)\}$. Thus, letting ϵ tend to zero yields (together with (3.7)) the desired result. ∎

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■

Proposition 3.1. *For a PRS, if either there exists an $\epsilon > 0$ such that $P^0(T_1 > \epsilon) = 1$, or the cycles $\{X_n\}$ form a k-dependent process, then (3.1) holds for all $g \in L_1^+(\pi)$ such that $E \int_0^{\tau(0)} g(\theta_s X) ds < \infty$.*

Proof: From Theorem 3.1, it suffices to show that $\{\frac{1}{t} \int_0^t g(\theta_s \circ X) ds : t \geq 0\}$ is uniformly integrable under P^0. If $P^0(T_1 > \epsilon) = 1$ then $\frac{1}{t} \int_0^t g(\theta_s \circ X) ds \leq \frac{1}{t} \sum_1^{[t/\epsilon]} J_k(g)$ which is UI for $g \in L_1^+(\pi)$. Now suppose that the cycles are k-dependent (in particular, X is ergodic). Then

$$\int_0^t g(\theta_s \circ X) ds \leq \sum_{j=1}^{N(t)+k} J_j(g)$$

$$= \sum_{j=1}^k J_j(g) + \sum_{j=k+1}^\infty J_j(g) I(N(t) \geq j - k). \tag{3.6}$$

By the assumption of k-dependency, the indicator $I(N(t) \geq j - k)$ is independent of $J_j(g)$ and hence taking expectations in (3.6) yields

$$E^0 \int_0^t g(\theta_s \circ X) ds \leq (E^0 N(t) + k) E^0 J_1(g). \tag{3.7}$$

By Proposition ? 1, $\{N(t)/t\}$ is UI and hence by (3.7) so is $\{\frac{1}{t} \int_0^t g(\theta_s \circ X) ds : t \geq 0\}$. ■

Remark(3.1): If X is null recurrent and non-ergodic, it is still possible that π as defined in (A.3) is a probability measure. In this case Theorem 3.1 remains valid. Take for example, a mixture of Poisson processes: Choose a r.v. Y such that $P(0 < Y \leq 1) = 1$ and $E(1/Y) = \infty$. Given Y, let $\{\tau(k)\}$ be a (non-delayed) Poisson process at rate Y. Define $X(t)$ as the forward recurrence time of this point process. Then the invariant σ-field is precisely $\sigma(Y)$, $E(T_1|Y) = 1/Y$ and hence $E(T_1) = \infty$. Moreover, given Y, the (marginal) steady-state distribution of $X(t)$ is exponential at rate Y. Thus the (unconditional) steady-state distribution is given by $F(x) = 1 - E(e^{-Yx})$.

Remark(3.2): The condition $\frac{1}{t} E \int_0^{t \wedge \tau(0)} g(\theta_s X) ds \longrightarrow 0$ is equivalent to UI of $\{\frac{1}{t} \int_0^{t \wedge \tau(0)} g(\theta_s X) ds\}$.

4. Continuous time Harris recurrent Markov processes

In this section we establish uniform limit theorems for continuous time Harris recurrent Markov processes (HRMP's) analogous to those already known (in the literature) to be true for discrete time Harris recurrent

Markov processes, called Harris recurrent Markov chains (HRMC's). Although renewal theory can be used to analyze HRMC's, the same is not true for HRMP's (as defined below).

Let $\{Z(t) : t \geq 0\}$ denote a Markov process with Polish state space S and paths in $\mathcal{D}_S$. We shall always assume that Z has the strong Markov property.

Z is called *Harris Recurrent* if there exists a non-trivial σ-finite measure μ on the Borel sets of S such that for any Borel set $A \subset S$

$$\mu(A) > 0 \Rightarrow P_z\left(\int_0^\infty 1_A \circ Z(t)dt = \infty\right) = 1 \qquad \text{for all} \quad z. \tag{4.1}$$

It is known that a HRMP has a unique invariant measure (up to multiplicative constant); see for example, [2] and [10]. If the invariant measure is finite then it is normalized to a probability measure in which case Z is called *positive recurrent*. In Theorem 2 of [10], it is proved that a Markov process Z is a positive HRMP if and only if it is a positive recurrent one-dependent regenerative (od-R) process, that is, an ergodic synchronous process with one dependent cycles. In particular, Corollary 3.1 and Proposition 3.1 both apply to positive HRMP's. So, for example, given any initial state $Z_0 = z$, it follows that the Cesaro averaged measures $\overline{\mu}_t^z(A) \overset{\text{def}}{=} \frac{1}{t}\int_0^t E_z I_A \circ (\theta_s Z)ds$ converge to π in total variation as $t \longrightarrow \infty$.

Once the od-R points have been selected for a HRMP, a natural question arises as to wether or not, by placing some regularity conditions (non-lattice (or spread-out) cycle length distribution, etc.) on the cycles of an HRMP Z, the *unaveraged* distributions will converge weakly (or, even better, in total variation) to π, that is, if $\mu_t^z(f) \overset{\text{def}}{=} E_z(f(\theta_t Z)) \longrightarrow \pi(f)$ for all bounded continuous f. The answer is no; a counterexample is given in Remark(3.2) of [10]. Also see example(1) of [4]. (It is true, however, that for each z, $\{\mu_t^z : t \geq 0\}$ is a tight collection of measures (see Theorem 2.1 of [4])).

Continuing in the spirit of Cesaro convergence we have

Proposition 4.1. *If Z is a positive HRMP with stationary distribution π then for each $g \in L_1^+(\pi)$,*

$$\sup_{|f| \leq g} |\overline{\mu}_t^z(f) - \pi(f)| \longrightarrow 0 \quad \text{for almost every } z \text{ w.r.t.} \quad \pi. \tag{4.1}$$

Proof: Let $r(z) \overset{\text{def}}{=} E_z \int_0^{\tau(0)} g(\theta_s X)ds$ and $\mathcal{E} \overset{\text{def}}{=} \{z : r(z) < \infty\}$. From Proposition 3.1 it suffices to show that $\pi(\mathcal{E}) = 1$. Now,

$$\pi(\mathcal{E}) = \lambda E^0 \int_0^{\tau(1)} I(r(Z(s) < \infty)ds. \tag{4.2}$$

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Moreover.

$$E^0\{r(Z(s);\tau(1) > s\} = E^0\{E^0_{Z(s)}\{\int_0^{\tau(1)} (g(\theta_u Z)du\};\tau(1) > s)\}$$

$$= E^0\{\int_s^{\tau(1)} (g(\theta_u Z)du \quad ;\tau(1) > s)\} \qquad .$$

$$\leq E^0\{\int_0^{\tau(1)} (g(\theta_u Z)du\} < \infty$$

Thus $r(Z(s)I(\tau(1) > s)) < \infty \quad P^0 \quad a.s.$ and hence $P^0\{r(Z(s)) < \infty;\tau(1) > s\} = 1$ for all $s \geq 0$. Integrating over s yields the result. ∎

Proposition 4.2. *If Z is a positive HRMP with stationary distribution π then*

$$\int \pi(dz) \sup_{|f| \leq g} |\overline{\mu}^z_t(f) - \pi(f)| \longrightarrow 0. \tag{4.3}$$

for all $g \in L^+_1(\pi)$ such that $E^0\{\int_0^{\tau(1)} u \; (\theta_u X)du\} < \infty$.

Proof: We must show that the error bounds for $|\overline{\mu}^z_t(f) - \pi(f)|$ can be integrated over z with respect to π. From the bounds obtained in (3.5)-(3.9) it suffices to show that $h(z) \stackrel{\text{def}}{=} E_z \int_0^{\tau(0)} g(\theta_s X)ds$ is in $L_1(\pi)$. An easy calculation yields

$$\int \pi(dz)h(z) = \lambda E^0 \int_0^{\tau(1)} \theta_s \circ \int_0^{\tau(1)} g(\theta_u X)duds$$

$$= \lambda E^0 \int_0^{\tau(1)} \int_s^{\tau(1)} g(\theta_u X)duds \qquad .$$

$$= \lambda E^0 \int_0^{\tau(1)} ug(\theta_u X)du$$

∎

Remark(4.1): In the proof of Proposition 4.1, the assertion that $\pi(\mathcal{E}) = 1$ amounts, in the terminology of discrete time Markov chain theory, to showing that a.e. state z (with respect to π) is $g - regular$ (see Proposition 5.13 of Nummelin[6]). In fact, Proposition 4.1 can be viewed as a continuous time Cesaro- average analog of Corollary 6.7i) in [6].

Remark(4.2): In Asmussen [1] the definition of HRMP is different than ours. Ours comes from Azema, Duflo and Revuz [2]. Asmussen's definition is more restrictive and in partic- ular implies the existence of an embedded renewal process.

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5. Applications to queues

In [9] the stability of open Jackson queueing networks is established where service times are i.i.d. with general distribution, exogenous interarrival times are i.i.d. with general distribution, and the routing is Markovian. We present here some immediate consequences of section 4 in the context of the above stability result.

Consider a c node queueing network with the nth exogenous customer (denoted by C_n) arriving at time t_n with $0 \leq t_1 \leq t_2 \leq \cdots$ and $\lim_{n \to \infty} t_n = \infty$. Each node is a FIFO single server station (with unlimited size waiting room). Upon arrival, each customer is assigned (independent of the past) an initial station according to the initial distribution $\mathcal{P} = (p_1, p_2, \ldots, p_c)$. Routing is *Markovian* : after completing service at node i, a customer is routed (independent of all else) to the end of the queue at node j with probability $r_{i,j}, (1 \leq i, j \leq c)$. In addition, we let $r_{i,g}$ denote the probability of leaving the system after a service completion at node i (and *going home*). Thus each customer's sequence of routings forms a Markov chain (with initial distribution $\mathcal{P}$) which we assume has precisely one set of absorbing states, the singleton $\{g\}$. $R = (r_{i,j})$ is called the *routing matrix*. Service times at the ith node $\{S_k(i) : k \geq 0\}$ are handed out by the server and assumed i.i.d. with distribution G_i and mean $0 \leq 1/\mu_i < \infty$; $S(i)$ will denote a generic service time $\sim G_i$. We assume that exogenous arrival epochs (t_n) form a renewal process with rate $0 \leq \lambda < \infty$ and (i.i.d.) interarrival times $T_n = t_{n+1} - t_n$; $E(T_n) = 1/\lambda$. T will denote a generic interarrival time. We let I_n denote the initial node for C_n; (I_n) forms an i.i.d. sequence with distribution $\mathcal{P}$. The service time sequences, the interarrival time sequence, and the initial node sequence are assumed independent. This is called a Jackson open network (JON) with general distribution i.i.d. input.

Let $Q_n = (Q_n(1), Q_n(2), \ldots, Q_n(c))$ denote the queue lengths (not including those in service) at the c nodes at time t_n- and $Y_n = (Y_n(1), Y_n(2), \ldots, Y_n(c))$ the residual service times (set to zero if server is free). It easily follows (by all the i.i.d. assumptions) that the process $X_n = (Q_n, Y_n)$ forms a Markov chain with state space $\mathcal{X} = N_+^c \times R_+^c$. Let $X(t) = (Q(t), Y(t))$ denote the queue length vector and residual service time vector at time t; $X_n = X(t_n-)$. Let $B(t)$ denote the forward recurrence time of the renewal process of exogenous arrivals; $B(t)$ is the time until the next arrival after time t. It is easily seen that $Z(t) = (X(t), B(t))$ is a Markov process with state space $\mathcal{S} = \mathcal{X} \times R_+$ and paths in $\mathcal{D}$.

For each i $(1 \leq i \leq c)$, let $N_n(i)$ denote the (random) number of times that C_n *will desire* service at

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node i. $M_i \stackrel{\text{def}}{=} E(N_n(i)) < \infty$ (in fact has finite moments of all orders). Define $\lambda_i \stackrel{\text{def}}{=} \lambda M_i$; we refer to λ_i as the *total arrival rate to node i*. It is well known that the λ_i's are the unique solution to the following set of c equations :

$$\lambda_i = \lambda p_i + \sum_{j \neq i} \lambda_j r_{ji} + \lambda_i r_{ii}, \quad 1 \le i \le c. \tag{2.1}$$

Finally, we define $\rho_i \stackrel{\text{def}}{=} \lambda_i / \mu_i$; it represents the long run average rate at which work arrives exogenously to the system *destined for node i*; $\rho \stackrel{\text{def}}{=} \rho_1 + \cdots + \rho_c$ denotes the total long run average rate at which work arrives exogenously to the system.

The following two theorems are proved in [9]:

Theorem 5.1. *The Markov chain $X = \{X_n\}$ for JON is Harris ergodic if $\rho_i < 1$ for each i ($1 \le i \le c$).*

Theorem 5.2 *If $\lambda > 0$ (i.e. interarrival times have finite first moment) then the Markov process Z for JON is a positive Harris recurrent Markov process (HRMP) if $\rho_i < 1$ for each i ($1 \le i \le c$). In particular it is positive recurrent one-dependent regenerative (od-R) with a unique steady-state distribution π. Moreover, $Z(t)$ converges to π in total variation if and only if A is spread-out. (In general, $\pi\{(X(t) = 0\} = 0$ and hence the regeneration points of Z are not described by consecutive visits of X to the empty state).*

From the above theorem we see that Z is an ergodic PRS with one-dependent cycles. Thus, so is any continuous functional $f(Z(t))$ such as total queue length $Q_T(t)$ (sum of the c queue lengths). Moreover, *total work in system* $w(t)$ is also; $w(t)$ denotes the sum of all remaining service times of all customers in the system (including their feedback) at time t (see section 4 of [9]). We thus obtain the following special cases of the results in section 4:

Proposition 5.1. *For a JON with $\rho_i < 1$ for each i ($1 \le i \le c$), the following hold:*

$$\frac{1}{t} \int_0^t P_z(w(s) \in \cdot) ds \longrightarrow P_\pi(w(0) \in \cdot) \quad \text{in total variation for each } z.$$

$$\frac{1}{t} \int_0^t P_z(Q_T(s) \in \cdot) ds \longrightarrow P_\pi(Q_T(0) \in \cdot) \quad \text{in total variation for each } z.$$

If $E_\pi(w(0)) < \infty$ then

$$\frac{1}{t} \int_0^t E_z(w(s)) ds \longrightarrow E_\pi(w(0)) \quad \text{for almost every } z \text{ w.r.t.} \pi.$$

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$$\frac{1}{t} \int_0^t E_z(Q_T(s))ds \longrightarrow E_\pi(Q_T(0)) \quad \text{for almost every } z \text{ w.r.t. } \pi.$$

APPENDIX : A brief introduction to synchronous processes

Our use of the word *synchronous* is from [3]. Other names have been given to a synchronous process; for example Serfozo [8] refers to them as semi-stationary processes. In Rolski[7] they arise as *Palm* versions of stationary processes (associated with point processes). Closely related to this is the general theory of stationary marked point processes. In any case, the ergodic properties of synchronous processes are well known in the literature. We state several such results the proofs of which can be found in, for example [3], [4], [5], [7] and [8].

Let $\theta_t : \mathcal{D} \to \mathcal{D}$ denote the *shift operator* $(\theta_t x)(s) = x(t+s)$.

Theorem A.1. *Suppose X is a PRS and $f : \mathcal{D}_S \to \Re$ is measurable. Let $J_n = J_n(f) \overset{\text{def}}{=} \int_{\tau(n-1)}^{\tau(n)} f(\theta_t \circ X)dt$. If $J_0(|f|) < \infty$ a.s. and if either $f \geq 0$ a.s. or $E\{J_1(|f|)\} < \infty$ then*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(\theta_s \circ X)ds = \frac{E\{J_1 | \mathcal{I}\}}{E\{T_1 | \mathcal{I}\}} \quad \text{a.s.} \tag{A.1}$$

where $\mathcal{I}$ denotes the invariant σ-field associated with $\{(X_n, T_n)\}$.

Let P^0 denote the probability measure under which X is non-delayed, that is, $P^0(X \in A) = P(\theta_{\tau(1)} \circ X \in A)$.

Corollary A.1. *Under the conditions of Theorem A.1, if in addition $\mathcal{I}$ is trivial (every set has probability 0 or 1) then $\{J_n, T_n : n \geq 1\}$ is ergodic and hence a.s.*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(\theta_s \circ X)ds = \frac{E\{J_1\}}{E\{T_1\}} = \lambda \int_0^\infty P^0(\theta_s \circ X \in A; \tau(1) > s)ds. \tag{A.2}$$

Under these circumstances, X is called ergodic.

The following Corollary follows from (A.1) by an elementary application of Fubini's Theorem and the Bounded Convergence Theorem.

Corollary A.2. *Under the hypothesis of Theorem A.1, if in addition f is bounded then*

$$\bar{\mu}_t(f) \overset{\text{def}}{=} \frac{1}{t} \int_0^t E f(\theta_s \circ X)ds \longrightarrow \pi(f) \overset{\text{def}}{=} E\left\{ \frac{E\{J_1 | \mathcal{I}\}}{E\{T_1 | \mathcal{I}\}} \right\}. \tag{A.3}$$

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π above defines a measure on $\mathcal{D}$ and (for reasons given below in Proposition A.1) is called the stationary probability measure for X. In particular, by choosing $f = 1_A$ (an indicator function), we have $\overline{\mu}_t(A) \to \pi(A)$ for each Borel set A of $\mathcal{D}$; thus the Cesaro averaged distributions converge weakly.

Proposition A.1. *Let π be the stationary measure of a PRS X. Then under π, $\theta = (\theta_s)$ is measure preserving on $\mathcal{D}$, that is, for each Borel set A, $\pi(A) = \pi(\theta_{-s}A)$ for all $s \geq 0$. In particular, if X has distribution π, then X is time stationary, that is, $\theta_t X$ has the same distribution for each $t \geq 0$.*

Let P^* denote the probability measure under which X has distribution π, that is, $P^*(X \in A) = \pi(A)$. From (A.2) we obtain for an *ergodic* synchronous process that

$$P^*(X \in A) = \lambda \int_0^\infty P^0(\theta_s \circ X \in A; \tau(1) > s)ds. \qquad (A.4)$$

If X is positive recurrent but not ergodic then the RHS of (A.4) still defines a probability measure on $\mathcal{D}$ (but not necessarily the same as the π from (A.3)). In fact, more can be said:

Proposition A.2 *For a PRS the RHS of (A.4) defines a probability measure on $\mathcal{D}$ (in general, not the same as π) under which $\theta = (\theta_s)$ is measure preserving.*

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Uniform Limit Theorems For Synchronous Processes
With Applications To Queues

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Karl Sigman [2]

Abstract

Let $X = \{X(t) : t \geq 0\}$ be a positive recurrent synchronous process (PRS), that is, a process for which there exists an increasing sequence of random times $\tau = \{\tau(k)\}$ such that for each k the distribution of $\theta_{\tau(k)}X \stackrel{def}{=} \{X(t + \tau(k)) : t \geq 0\}$ is the same and the cycle lengths $T_n \stackrel{def}{=} \tau(n+1) - \tau(n)$ have finite first moment. In the present paper we investigate conditions under which the Cesaro averaged functionals $\Pi_t(f) \stackrel{def}{=} \frac{1}{t}\int_0^t E(f(\theta_s X))ds$ converge uniformly (over a class of functions) to $\pi(f)$, where π is the stationary distribution of X. We show that $\Pi_t(f) \longrightarrow \pi(f)$ uniformly over f satisfying $\|f\|_\infty \leq 1$ (total variation convergence). We also show that to obtain uniform convergence over all f satisfying $|f| \leq g$ ($g \in L_1^+(\pi)$ fixed) requires placing further conditions on the (PRS.) This is in sharp contrast to both classical regenerative processes and discrete time Harris recurrent Markov chains (where renewal theory can be applied) where such uniform convergence holds without any further conditions. For continuous time positive Harris recurrent Markov processes (where renewal theory can not be applied) we show that these further conditions are in fact automatically satisfied. In this context, applications to queueing models are given.